Delta method

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① 已知 EX, $\text{var}\,X$, 求 $E(g(X))$, $\text{Var}(g(X))$

$g(X)$ 在 $X = \mu_X$ 处的 Taylor expansion.

$$g(x) = g(x_0) + (x-x_0)g'(x_0) + (x-x_0)^2\frac{g''(x_0)}{2!} + (x-x_0)^3\frac{g^{(3)}(x_0)}{3!}$$

μ_X处展开

$$g(x) = g(\mu_X) + (x-\mu_X)g'(\mu_X) + (x-\mu_X)^2\frac{g''(\mu_X)}{2!} + \frac{(x-\mu_X)^3 g^{(3)}(\mu_X)}{3!}$$

$$E[g(x)] = g(\mu_X) + \underbrace{E(X-\mu_X)}_{EX-\mu_X = \mu_X-\mu_X = 0}g'(\mu_X) + E(X-\mu_X)^2\frac{g''(\mu_X)}{2} + \text{Remainder}$$

☆ $$E(g(x)) \approx g(\mu_X) + \text{Var}(x)\frac{g^{(2)}(\mu_X)}{2}$$

→ special case $g(x) = ax+b$, $g^{(2)}(\mu_X) = 0$.
$E(gX) = g(\mu_X) = g(EX)$

$$\text{Var}(g(x)) \simeq \underbrace{\text{Var}(g(\mu_X))}_{\text{Var(constant)}=0} + \text{Var}\left((x-\mu_X)g'(\mu_X)\right) + \cancel{\text{Var}(x-\mu_X)^2\frac{g''(\mu_X)}{2!}}$$

☆ $$\approx [g'(\mu_X)]^2 \cdot \text{Var}\,X.$$

$$E(g(x)) \simeq g(\mu_X) + \frac{\text{Var}(x)}{2}\cdot g^{(2)}(\mu_X)$$

→ 用 $g(\mu_X)$ 去 approx. 这一项是 correction term.

if $g''(\mu_X) < 0$ concave, $E(g(x)) < g(\mu_X) = g(EX)$

if $g''(\mu_X) > 0$ convex, $E(g(x)) > g(EX)$

Jensen's Inequality Horray !!

② Generalization of CLT (w.r.t. sample mean): the Delta method.

$\sqrt{n}(Y_n - \theta) \xrightarrow{D} (0, \sigma^2)$, give $g(y_n)$

we have:

$$\sqrt{n}(g(y_n) - g(\theta)) \xrightarrow{D} (0, \sigma^2[g^{(2)}(\theta)]^2)$$

proof. $g(y_n)$ 在 $y_n = \theta$ 处的展开.

$$g(y_n) = g(\theta) + (y_n-\theta)g'(\theta) + \text{Remainder}$$

$$g(y_n) - g(\theta) = g'(\theta)(y_n-\theta)$$

$$\sqrt{n}(g(y_n) - g(\theta)) = \sqrt{n}(y_n-\theta)g'(\theta)$$

LHS: $\sqrt{n}(y_n-\theta) \xrightarrow{D} N(0, \sigma^2)$

by Slutsky's theorem,

$$\sqrt{n}(y_n-\theta)\cdot g'(\theta) \xrightarrow{D} N(0, \sigma^2[g'(\theta)]^2)$$

it then follows that:

$$\sqrt{n}(g(y_n) - g(\theta)) \xrightarrow{D} N(0, \sigma^2[g'(\theta)]^2)$$

③ extension if $g'(\theta) = 0$, take one more term in taylor expansion.

$$g(y_n) = g(\theta) + (y_n-\theta)g'(\theta) + (y_n-\theta)^2\frac{g^{(2)}(\theta)}{2!}$$

$$g(y_n) - g(\theta) = (y_n-\theta)^2\frac{g^{(2)}(\theta)}{2}$$

$$\frac{n(g(y_n) - g(\theta))}{\sigma^2} = \frac{n(y_n-\theta)^2}{\sigma^2}\cdot\frac{g^{(2)}(\theta)}{2}$$

凑一个 $\chi^2(1)$

$$\sqrt{n}(y_n-\theta) \sim N(0, \sigma^2)$$

$$\frac{\sqrt{n}(y_n-\theta)}{\sigma} \sim N(0, 1)$$

$$\therefore \left[\frac{\sqrt{n}(y_n-\theta)}{\sigma}\right]^2 = \frac{n(y_n-\theta)^2}{\sigma^2} \sim \chi^2(1)$$

$$\therefore \frac{n(g(y_n) - g(\theta))}{\sigma^2} \sim \chi^2(1)\cdot\frac{g^{(2)}(\theta)}{2}$$

$$n[g(y_n) - g(\theta)] \sim \chi^2(1)\frac{g^{(2)}(\theta)}{2}\cdot\sigma^2$$

④ extension: multivariate case

$$\sqrt{n}[g(\bar{X}_1, \bar{X}_2 \ldots \bar{X}_s) - g(\mu_1, \mu_2 \ldots \mu_s)] \xrightarrow{D} N(0, \tau^2)$$

$$\tau^2 = \sum\sum\sigma_{ij}\frac{\partial g(\mu)}{\partial\mu_i}\frac{\partial g(\mu)}{\partial\mu_j}, \quad \sigma_{ij} = \text{cov}(X_{ik}, X_{jk})$$

$E(X_{ij}) = \mu_i$